In order to obtain a bound on $\bar{x}(\theta_0)$, Eq. (9) can be solved to give

$$\bar{x}(\theta_0) = \frac{\Delta C}{C_0}\theta_0 - 2\int_0^{\theta_0} \bar{y}(\tau)d\tau \quad (19)$$

and by means of the triangle inequality

$$\theta_0 | \,|(\Delta C/C_0)| - 2|\bar{y}(\theta_0)|\,| \leqslant |\bar{x}(\theta_0)| \leqslant \theta_0[\,|(\Delta C/C_0)| + 2|\bar{y}(\theta_0)|\,] \quad (20)$$

where $|\bar{y}(\theta_0)|$ is the upper bound obtained in (16). If the upper bound (16) is close to the actual value of $|\bar{y}(\theta)|$, then the lower bound in (20) would give a close approximation to $|\bar{x}(\theta_0)|$. The bound on $|x(\theta)|$ can be obtained by multiplication of (20) by $r_0(\theta_0)$.

The upper bound on $y(\theta_0)$ as given by (16) was found, in some cases, to be more accurate than that obtained from the averaged equations for it produced lower magnitudes. However, the bound on $x(\theta_0)$ was less effective than that of the averaged equations. The reason for the latter discrepancy is that the procedure in this Note produces accurate bounds on $|d\bar{x}/d\theta_0|$, from which bounds on $|\bar{x}(\theta_0)|$ are deduced by a straightforward integration. If $\bar{y}(\theta_0)$ or the variations in $\bar{x}(\theta_0)$ changes sign, the bound on $|\bar{x}(\theta_0)|$ becomes ineffective.

References

[1] Frick, R. H., "Stability of a Cloud of Orbiting Dipoles," RM-3296, Aug. 1962, The RAND Corp., Santa Monica, Calif.

[2] Bellman, R., *Perturbation Techniques in Mathematics*, Holt, Rinehart & Winston, New York, 1966, p. 89.

Correct Formulation of Airfoil Problems in Magnetoaerodynamics

L. Dragos*
Faculty of Mathematics and Mechanics, Bucharest, Rumania

1. Introduction

In a recent paper[1] Fan and Ludford have dealt with the theory of thin airfoils in magnetoaerodynamics in view of giving a correct solution to this problem. It may be easily observed, however, that the kernels of the integral equations to which the solution of the problem is reduced are expressed by divergent integrals (e.g., $I_{\pm}$). Under such conditions the solution given by the mentioned authors cannot be valid. The same incorrectness is also found in Ref. 2.

The purpose of this Note is to revise the solution of this problem. The method of solution is based essentially on the ideas which have substantiated the first article on this problem[3] and then under a simpler form.[4]

This Note treats only the case of crossed fields, to which the paper by Fan and Ludford also refers.

2. Motion Equations

In dimensionless variables the system of motion equations may be written as follows:

$$M^2\,\partial p/\partial x + \partial v_x/\partial x + \partial v_y/\partial y = 0 \quad (1)$$

$$\partial v_x/\partial x + \partial p/\partial x = S(\partial b_x/\partial y - \partial b_y/\partial x) \quad (2)$$

$$\partial v_y/\partial x + \partial p/\partial y = 0 \quad (3)$$

$$\partial b_x/\partial x + \partial b_y/\partial y = 0 \quad (4)$$

$$\partial b_y/\partial x - \partial b_x/\partial y = R(b_y + v_x) \quad (5)$$

$$\lim_{x^2+y^2\to\infty} (v_x,v_y,p,b_x,b_y) = 0,\ S = A^{-2},\ R = Rm \quad (6)$$

Elimination of the pressure from (1) and (2) yields

$$\beta^2\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = SM^2\left(\frac{\partial b_y}{\partial x} - \frac{\partial b_x}{\partial y}\right),\ \beta^2 = 1 - M^2 \quad (7)$$

Taking account of (4) and (7) in the equation obtained by eliminating the pressure from (2) and (3) we find

$$Hv_y + S\Delta b_x = 0,\ H = \beta^2\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},\ \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad (8)$$

From (5) and (4) we get

$$(\Delta - R\,\partial/\partial x)b_x + R\,\partial v_x/\partial y = 0 \quad (9)$$

Deriving (7) with respect to y and taking account of (4) and (9), we obtain

$$\left[\beta^2\frac{\partial}{\partial x}\left(\Delta - R\frac{\partial}{\partial x}\right) - RSM^2\Delta\right]b_x = R\frac{\partial^2 v_y}{\partial y^2} \quad (10)$$

Finally, from (8) and (10) we obtain

$$L\binom{v_y}{b_x} = 0$$
$$L = H\frac{\partial}{\partial x}\left(\Delta - R\frac{\partial}{\partial x}\right) - RS\Delta\left(M^2\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right) \quad (11)$$

the operator L being the same as in previous papers.

3. Dispersion Equation

For plane waves of the form $\exp(-i\lambda x + sy)$, $s = -i\lambda r$, we get from (11) the following dispersion equation:

$$a(1 + r^2)^2 - b(1 + r^2) + c = 0$$
$$a = RS - i\lambda,\ b = R(1 + S + SM^2) - i\lambda M^2 \quad (12)$$
$$c = M^2R$$

The roots of Eq. (12) are distinct and have the imaginary part differing from zero. We denote by r_j ($j = 1,2$) those roots of Eq. (12) for which the real part of expressions $s_j = -i\lambda r_j$ are negative. This fact is possible since Eq. (12) is biquadratic such that two roots will surely enjoy this property indifferently if λ is positive or negative. The other two roots are of opposite sign. The roots of expression (12) are expressed by radicals.

Taking into account that if m and n are real numbers, we have

$$\left(m + \frac{n}{i\lambda}\right)^{1/2} = \begin{cases} (m)^{1/2} + \dfrac{n}{2(m)^{1/2}}\dfrac{1}{i\lambda} + 0(\lambda^{-2}),\ m > 0 \\ i(-m)^{1/2}\,\mathrm{sign}\lambda + \dfrac{n}{2(-m)^{1/2}}\dfrac{1}{|\lambda|} + 0(\lambda^{-2}),\ m < 0 \end{cases}$$

and we deduce that for large λ we have the following behavior:

$$r_1 = -\,i\,\mathrm{sign}\lambda + R0(|\lambda|^{-1})$$

$$r_2 = \begin{cases} -\,i\beta\,\mathrm{sign}\lambda + R0(|\lambda|^{-1}),\ \beta^2 > 0,\ \beta = (1 - M^2)^{1/2} \\ -(-\beta^2)^{1/2} + R0(\lambda^{-1}),\ \beta^2 < 0 \end{cases} \quad (13)$$

Received March 10, 1969.

* Professor, Str. Academiei 14.

4. General Solution

Taking account of (6), (3), (4), and (1) we get the following solution:

$$\begin{Bmatrix} v_y^{\pm} \\ b_x^{\pm} \\ p^{\pm} \\ b_y^{\pm} \\ v_x^{\pm} \end{Bmatrix} (x,y) = \int_{-\infty}^{+\infty} \sum_j \begin{Bmatrix} \mp r_j A_j^{\pm} \\ \mp r_j B_j^{\pm} \\ A_j^{\pm} \\ B_j^{\pm} \\ (r_j^2 - M^2)A_j^{\pm} \end{Bmatrix} \times \exp(-i\lambda x \pm s_j y)d\lambda \quad (14)$$

where, as usual, the upper sign indicates the solution valid in the upper half-plane ($y > 0$) and the lower sign the solution valid in the lower half-plane ($y < 0$).

The verification of Eq. (2) implies

$$(\beta^2 + r_j^2)A_j^{\pm} + S(1 + r_j^2)B_j^{\pm} = 0 \qquad (15)$$

5. Boundary Conditions

If we assume for the sake of simplicity that the airfoil equation is $y = Y(x)$, $|x| \leqslant 1$, we have

$$v_y^+(x,0) = Y'(x), \ |x| \leqslant 1 \qquad (16)$$

$$[v_y] = [b_x] = [b_y] = 0, \ \forall x \qquad (17)$$

with the notation

$$[\phi] = \phi^+(x,0) - \phi^-(x,0)$$

Using the general solution and the inversion theorem of the Fourier integrals, from (16) and (17) we deduce

$$\int_{-\infty}^{+\infty} \Sigma r_j A_j^+ \exp(-i\lambda x)d\lambda = -Y'(x), \ |x| \leq 1 \qquad (18)$$

$$\Sigma r_j(A_j^+ + A_j^-) = 0, \ \Sigma r_j(1 + r_j^2)^{-1}(A_j^+ + A_j^-) = 0 \qquad (19)$$

$$\Sigma(\beta^2 + r_j^2)(1 + r_j^2)^{-1}(A_j^+ - A_j^-) = 0 \qquad (20)$$

Taking into account that the roots r_j are distinct, from (19) we have

$$A_j^+ + A_j^- = 0 \qquad (21)$$

and then from (20)

$$A_2^+ = \omega A_1^+ \qquad (22)$$

$$\omega = -(\beta^2 + r_1^2)(1 + r_2^2)(\beta^2 + r_2^2)^{-1}(1 + r_1^2)^{-1} \qquad (22')$$

such that (18) becomes

$$\int_{-\infty}^{+\infty} (r_1 + \omega r_2)A_1^+ \exp(-i\lambda x)d\lambda = -Y'(x), \ |x| \leqslant 1 \qquad (23)$$

Using the pressure expression, we deduce

$$[p] = 2\int_{-\infty}^{+1} (1 + \omega)A_1^+ \exp(-i\lambda x)d\lambda = 2f(x) \qquad (24)$$

which is useful for the calculation of lift.

We shall consider the function $f(x)$ unknown. From the condition of pressure continuity outside the profile there results $f(x) \equiv 0$, $|x| > 1$. With the aid of the Fourier inversion theorem from (24) we get

$$(1 + \omega)A_1^+ = \frac{1}{2\pi}\int_{-1}^{+1} f(t)\exp(i\lambda t)dt \qquad (25)$$

such that (23) furnishes

$$\int_{-1}^{+1} f(t)K(t - x)dt = -Y'(x), \ |x| \leqslant 1 \qquad (26)$$

where

$$K(t - x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} k(\lambda)\exp\{i\lambda(t - x)\}d\lambda$$

$$k = \frac{M^2 r_1 r_2 - \beta^2(1 + r_1^2 + r_2^2) - r_1^2 r_2^2}{M^2(r_1 + r_2)} \qquad (26')$$

The integral equation (26) solves the problem. If f is determined, A_1^+ results from (25), A_2^+ results from (22), and A_j^- from (21). All determinations are possible owing to the fact that the roots r_j are distinct.

6. Examination of Convergence

Taking account of (13), we obtain

$$k = \begin{cases} -i\beta \operatorname{sign}\lambda + Rk_1, & M < 1 \\ -(-\beta^2)^{1/2} + Rk_2, & M > 1 \end{cases} \qquad (27)$$

where $k_1 = 0(\lambda^{-1})$ and $k_2 = 0(\lambda^{-1})$, which shows that the kernel (26′) is a distribution. Since[5]

$$\int_{-\infty}^{+\infty} \operatorname{sign}\lambda \exp\{i\lambda(t - x)\,d\lambda\} = 2i(t - x)^{-1}$$

$$\int_{-\infty}^{+\infty} \exp\{i\lambda(t - x)\}\,d\lambda = 2\pi\delta(t - x)$$

δ being Dirac's distribution, and since

$$\int_{-\infty}^{+\infty} f(t)\delta(t - x)dt = f(x)$$

Eq. (26) may be written as follows:

$$\frac{\beta}{\pi} P\int_{-1}^{+1}\frac{f(t)dt}{t - x} + R\int_{-1}^{+1} f(t)K_1(t - x)dt = -Y'(x), \ M < 1 \qquad (28)$$

$$(M^2 - 1)^{1/2}f(x) = R\int_{-1}^{+1} f(t)K_2(t - x)dt + Y'(x), \ M > 1 \qquad (29)$$

where

$$K_j = \frac{1}{2\pi}\int_{-\infty}^{+\infty} k_j \exp\{i\lambda(t - x)\}\,d\lambda, \ j = 1,2$$

are convergent integrals. In (28) the sign P signifies the principal value in the sense given by Cauchy. Accordingly, in the case of supersonic motions we are led to an integral equation of the Fredholm type (29), whose solution can be obtained by successive approximations.

In order to reduce Eq. (28) to an equation of the same type, we shall use the solution of the following singular equation:

$$\frac{1}{\pi} P\int_{-1}^{+1}\frac{f(t)dt}{t - x} = F(x) \qquad (30)$$

met with in the theory of wing in the classical aerodynamics. If F satisfies Hölder's condition, the solution of Eq. (30) is[6]

$$f(x) = -\frac{1}{\pi}\left(\frac{1 - x}{1 + x}\right)^{1/2} P\int_{-1}^{+1}\left(\frac{1 + t}{1 - t}\right)^{1/2}\frac{F(t)dt}{t - x} \qquad (30')$$

Accordingly, if Y' satisfies Hölder's condition, from (28) we obtain

$$\beta f(x) = R\int_{-1}^{+1} F(\zeta)N(\zeta,x)d\zeta + m(x), \ M < 1 \qquad (31)$$

where

$$N(\zeta,x) = \frac{1}{\pi}\left(\frac{1 + x}{1 + x}\right)^{1/2} P\int_{-1}^{+1}\left(\frac{1 + t}{1 - t}\right)^{1/2}\frac{K_1(\zeta - t)}{t - x}\,dt$$

$$n(x) = \frac{1}{\pi}\left(\frac{1 - x}{1 + x}\right)^{1/2} P\int_{-1}^{+1}\left(\frac{1 + t}{1 - t}\right)^{1/2}\frac{Y'(t)}{t - x}\,dt \qquad (31')$$

This is the final solution of the problem.

When the airfoil equation is $y = Y_{\pm}(x)$ the integral equations of the problem are the same, with the only difference that the last term in (29) and the last term in (31) contain the expressions $Y_{\pm}(x)$.

References

[1] Fan, D. N. and Ludford, G. S. S., "Correct Formulation of Airfoil Problems in Magnetoaerodynamics," *AIAA Journal*, Vol. 6, No. 1, Jan. 1968, pp. 167–169.

[2] Bhutani, O. P. and Nanda, K. D., "A General Theory of Thin Airfoils in Nonequilibrium Magnetogasdynamics. Part II: Transverse Magnetic Field," *AIAA Journal*, Vol. 6, No. 11, Nov. 1968, pp. 2122–2124.

[3] Dragos, L. "Theory of Thin Airfoils in Magnetoaerodynamics," *AIAA Journal*, Vol. 2, No. 7, July 1964, pp. 1223–1229.

[4] Dragos, L., "Reply by Author to P. Grenberg," *AIAA Journal*, Vol. 5, No. 7, July 1967, pp. 1367–1370.

[5] Guelfand, J. M. and Shilov, G. E., *Generalized Functions I*, Fizmatghiz, Moscow, 1958.

[6] Iacob, C., *Introduction Mathématique a la Mécanique des Fluides*, Bucarest E'ditions de l' Academie de la République Populaire Roumaine, Gauthier-Villars, Paris, 1959.

Bending of a Beam Made of a Fiber-Reinforced Viscoelastic Material

R. Ray Nachlinger* and J. R. Leininger†
University of Houston, Houston, Texas

Introduction

Due to their high strength and low weight, there has been a great increase in the use of fiber-reinforced materials in the past few years. Many of these materials consist of high-strength fibers imbedded in a matrix of a viscoelastic material.

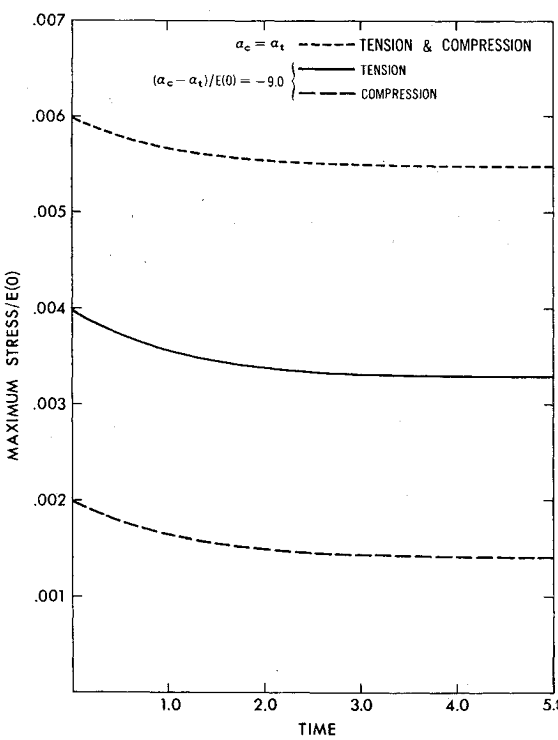

Fig. 1 Maximum stress vs time, constant radius of curvature, R = 1000.

Received April 1, 1969.

* Assistant Professor, Department of Mechanical Engineering.

† Department of Mechanical Engineering.

The long, thin fibers make the resulting composite material much stronger than the matrix alone in tension; but when the composite is compressed, the fibers buckle and a larger portion of the load is carried by the matrix. The net result of this effect is that if one has a composite with thin, long fibers, the material will behave essentially as an elastic material in tension and as a viscoelastic material in compression.

This Note is an investigation of the bending of a beam made of the type of material previously discussed. The analysis that is carried out is for the case of a beam that is subjected to pure moments. The results of this analysis are quite surprising. They show that the maximum stress in the beam can be 50% greater than that predicted if the material is assumed to have the same modulus in both tension and compression.

Development of the Equations

The material from which the fiber-reinforced material is made is assumed to have a stress-strain relation given by

$$\sigma(t) = [\alpha + E(0)]\epsilon(t) + \int_0^t \dot{E}(\tau)\epsilon(t - \tau)d\tau \qquad (1)$$

where $E(t)$ is the relaxation function of the matrix and α takes into account the added stiffness due to the fibers. This relation assumes a rigid bond between the matrix and the fibers. Since the fibers will buckle in compression, α will be smaller when the material is in compression than when it is in tension. Thus, α is taken to be

$$\alpha = \begin{cases} \alpha t \ \epsilon > 0 \\ \alpha c \ \epsilon < 0 \end{cases} \qquad (2)$$

For the bending of a beam, the relevant equilibrium equations are

$$\textstyle\int_A \sigma dA = 0, \int_A \sigma y dA = -M \qquad (3)$$

where y is measured from the neutral axis. The usual strain-curvature assumption will also be made, i. e.,

$$\epsilon = -y/\rho \qquad (4)$$

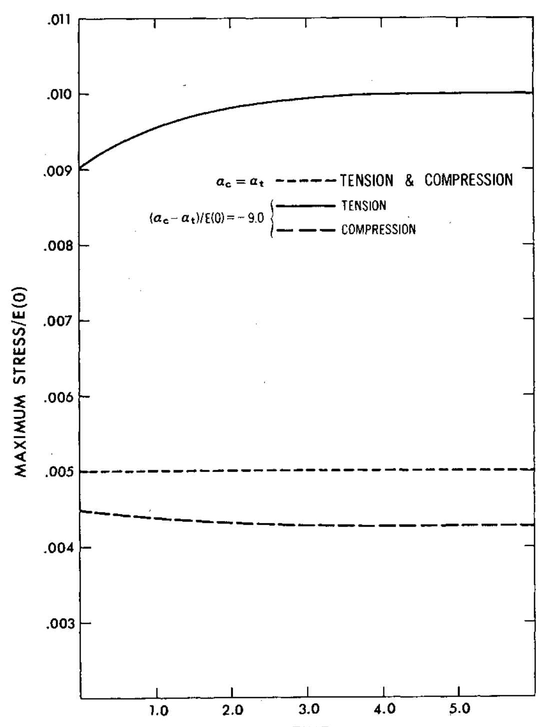

Fig. 2 Maximum stress vs time; $M/E(0)$ = 0.001, $(\alpha_c - \alpha_t)/E(0) = -9.0$.